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ESTIMATING PERFORMANCE CAPABILITIES OF BOOST ROCKETS

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ESTIMATING PERFORMANCE CAPABILITIES OF BOOST ROCKETS

This paper reports results of a parametric study of boost rockets. The term boost rocket includes rockets launched from the surface of the earth for the purpose of achieving near-orbital or greater velocities.

The parameters studied can be divided into two categories: vehicle design parameters and trajectory parameters. Vehicle design parameters describe the physical rocket and include such quantities as weights, thrusts, propellant flow rates, drag coefficients, and the like. A set of these parameters would serve as a basic set of specifications with which to design a vehicle. Trajectory parameters include such quantities as impact range, apogee altitude, and burnout velocity. Trajectory parameters can serve, though not uniquely, as specifications for a missile system as well. A particular vehicle system can perform many missions, and any one mission can be performed by many vehicles. We usually think of missions in terms of trajectory parameters and vehicles in terms of design parameters, and the problem becomes to relate the two.

The simplest relation is found in the well-known equation:

$$V_i = I_i \, g \ln r_i \tag{1}$$

where

I_i = stage i specific impulse; thrust divided by flow rate of fuel

g = gravitational constant = 32.2 ft/sec^2

r_i = stage i burnout mass ratio; initial mass divided by burnout mass

V_i = velocity added during stage i.

If several stages are used, the total velocity is the sum of the velocities added during each stage. Certain assumptions are used in the derivation of the rocket equation which limits its usefulness for boost rockets. They are: (a) no gravitational acceleration, (b) no drag, and (c) specific impulse is constant. When it becomes necessary to include these effects, the most frequent technique is to solve the differential equations of motion by use of a computing machine.

Since some of the inputs to the problem are not analytic, such as drag coefficient versus Mach number, the machine uses an integration technique which virtually "flies" the missile on the computer. In this manner impact range, apogee altitude, burnout velocity, burnout altitude, and so forth can be determined as functions of vehicle design parameters.

The same vehicle can be flown on many paths, so it is necessary to provide the machine with some sort of steering program. The most frequently used program for the atmospheric portion of flight is the "zero-lift" turn. Assuming that the rocket thrust vector is aligned with the vehicle longitudinal axis, the vehicle attitude is programmed to coincide with the rocket velocity vector. For this reason, the zero-lift trajectory is frequently referred to as the "gravity turn." If a rotating earth is used, the thrust is aligned with the velocity vector as computed in a rotating coordinate system. Since the missile is launched with zero initial velocity, a singularity exists for the velocity angle at the instant of launch. All gravity turn trajectories, regardless of burnout angle, must launch vertically. For that reason, a mathematical artifice (an initial "kick" angle) is applied to the velocity vector a few seconds after launch to start the turn.

Most problems can be solved by the computer very quickly, and the accuracy of the results is almost beyond question. But there are disadvantages as well. First, the actual computer time consumed may be small, but the time required to prepare the input data and arrange for computer time can be quite long in comparison. Secondly, the accuracy required of results for preliminary design purposes is quite different from that required for, let us say, targeting purposes; and the high accuracy offered by the digital machine frequently goes to waste. Finally, it is one thing to be able to feed the computer one set of data and have a set of answers returned and quite another to be able to view an analytic relation or graph and get a "feel" for the whole system. For these reasons, simplified, even if approximate, solutions to the problem of determining trajectory parameters for boost vehicles are quite useful.

Two techniques may be employed to determine approximate solutions to the differential equations of motion. One technique uses approximation <u>before</u> the

equations are solved. The original model is transformed into a simpler one for which the solutions are known. In this case one must make à priori guesses as to the accuracy lost in simplification. However, the digital computer has provided the tool for making approximations after solution. The model which is simplified is the solution, not the set of differential equations; and the accuracy of the approximations can be readily observed. The latter technique has been employed in this study.

The differential equations are helpful in showing which variable will be important to consider. A short, theoretical analysis, appended to this report, has shown that the following missile design parameters, together with a burnout velocity angle, will determine a trajectory.

I	vacuum specific impulse; vacuum thrust divided by flow rate
r	mass ratio
N_o	ratio of initial (launch) thrust to lift-off weight
$\frac{C_{DM}A}{W_o}$	drag parameter; C_{DM} is the maximum value for drag coefficient versus Mach number, A is the reference area, and W_o is the lift-off weight of missile.
$\frac{I_s}{I}$	ratio of initial specific impulse to vacuum specific impulse
t_b	burning time $= \frac{I_s}{N_o}\left(1 - \frac{1}{r}\right)$ for constant weight flow rate

The trajectory parameters studied are:

V_b	burnout velocity
β_b	velocity burnout angle (with respect to local vertical)
h_b	burnout altitude from the earth's surface
x_b	surface range at burnout
R	impact range

It is clear from the number of parameters studied that it would be impossible to simply plot the results. Therefore, simplification and codification of the results have been a significant part of the study. Results are presented in two forms: (a) a set of general equations for determining V_b, h_b, and x_b versus β_b for selected ranges of missile design parameters (where necessary "constants" used in the equations are presented in graphical form) and (b) a simple equation for maximum impact range as a function of missile parameters, together with many of its derivatives.

In addition, a table of equations of several free-flight trajectory parameters based on the Kepler ellipse is included. These equations are well known but are included for convenience. These formulas, together with burnout conditions determined from the computer study, will aid in the solution of a large variety of the problems frequently encountered in preliminary design.

The free-flight trajectory for a vehicle is defined by the velocity and position vectors at burnout. The velocity vector is defined in terms of its magnitude[1] V_b and its angle with respect to the local vertical β_b. The position vector is defined by an altitude h and surface range x_b. V_b, h_b, and x_b are determined as functions of β_b and the vehicle design parameters.

VELOCITY VERSUS BURNOUT ANGLE

Using Equation (1) the "theoretical" burnout velocity may be determined for a vehicle. We define the quantity V_L as being the loss in velocity caused by gravitation and atmosphere.

$$V_b = V^* - V_L \tag{2}$$

where

$$V^* = \Sigma V_i = \Sigma I_i g \ln r_i \tag{3}$$

[1] The term velocity will refer to the magnitude of the velocity vector. If the vector is meant, velocity vector will be used.

An empirical equation for V_L in terms of vehicle design parameters has been derived by comparing results of several hundred machine trajectory calculations assuming single-stage vehicles, a gravity turn, and a spherical, non-rotating earth.

$$V_L = (gt_b - K_{gg}) \left\{ 1 - K_g \left(1 - \frac{1}{r}\right) \left(\frac{\beta_b}{90°}\right)^2 \right\} + K_D \frac{C_{DM}A}{W_o} + K_a \qquad (4)$$

It will be convenient to discuss the equation term by term, so we will designate the three components as follows:

$$V_g = \text{gravitational loss} = (gt_b - K_{gg}) \left\{ 1 - K_g \left(1 - \frac{1}{r}\right) \left(\frac{\beta_b}{90°}\right)^2 \right\} \qquad (5)$$

$$V_D = \text{drag loss} = K_D \frac{C_{DM}A}{W_o} \qquad (6)$$

$$V_a = \text{nozzle-pressure loss} = K_a \qquad (7)$$

Gravitational Loss

The gravitational loss was determined by setting the drag equal to zero and flying the vehicle to several burnout angles. The term gt_b is the gravitational loss to be expected from a vertical flight in a constant gravitational field. A realistic gravitational field varies in an inverse square of the distance from the earth's center, so the term actually overestimates this loss. For ranges of vehicles using currently available propellants, the differences between the amount gt_b and the correct gravity loss will be small; and for this equation the difference has been included as the constant K_{gg}. The term $\left\{ 1 - K_g \left(1 - \frac{1}{r}\right) \left(\frac{\beta_b}{90}\right)^2 \right\}$ fits a curve as a function of β_b. The constant K_g was determined by a least-squares curve-fitting technique and usually resulted in a curve fit which was within 30 feet per second of the machine results. The above form was found to fit actual results better than a more obvious choice, $K \cos \beta_b$. The latter resulted in maximum differences of 300 feet per second. Curves for K_g versus I and N_o are found in Figure 1, and a curve for K_{gg} versus I is found in Figure 2.

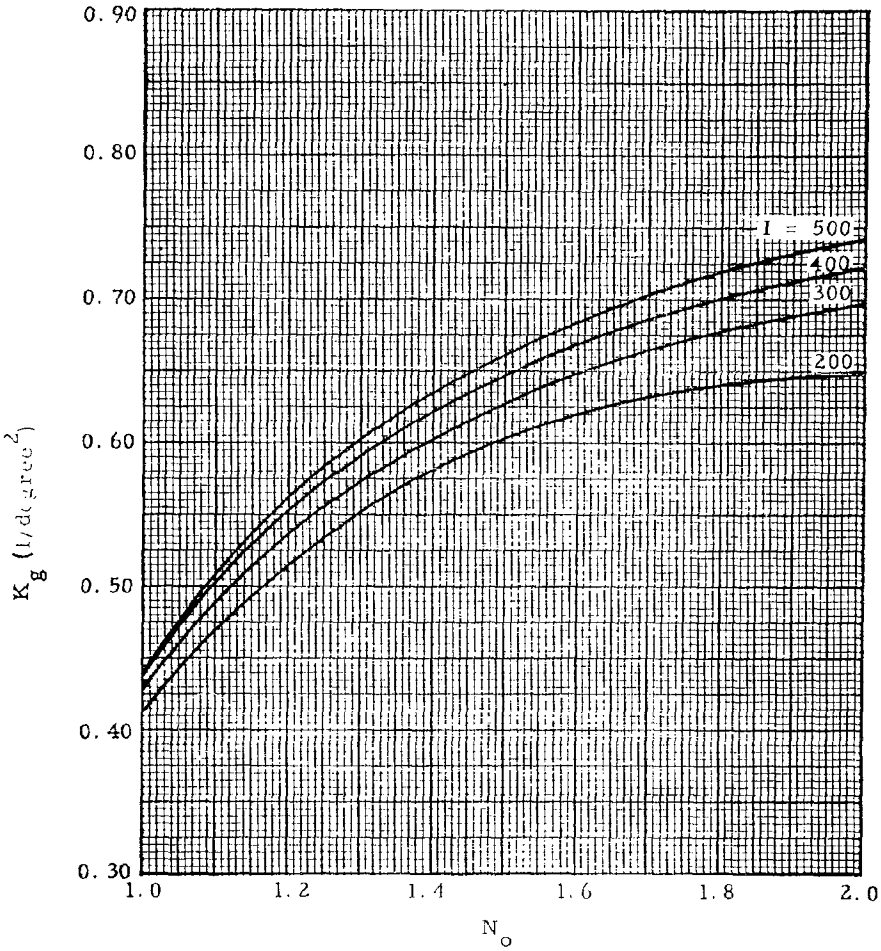

Figure 1. K_g Versus Vacuum Specific Impulse, I, and Initial Thrust-to-Weight Ratio, N_o.

(2009)

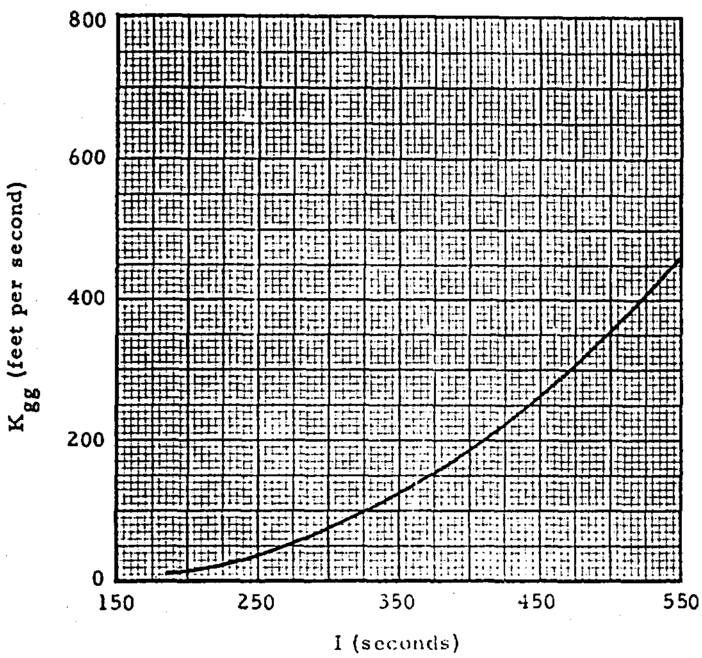

Figure 2. K_{gg} Versus Vacuum Specific Impulse, I.

(2015)

Drag Velocity Loss

The velocity lost to drag is proportional to the quantity $C_{DM}A/W_o$. C_{DM} has been chosen as a single parameter to define all drag curves. The reasons for this choice are: (a) that most realistic drag curves have approximately the same form, except for the absolute magnitudes of the values, and (b) that the greater portion of the drag loss occurs early in powered flight, where C_D attains a maximum. The actual drag curve used in the machine trajectory calculation is shown in Figure 3. The empirical constant K_D was obtained by computing the difference between burnout velocities for similar vehicles with and without drag. All comparisons were made for identical burnout angles. The constant was found to be a function of I_s/N_o, β_b, and N_o. However, K_D was so weakly dependent upon N_o that this effect was disregarded for simplicity in presenting the results. K_D is shown in Figure 4 as a function of I_o/N_o and β_b.

Nozzle-Pressure Loss

For the same propellant flow rate, the effective thrust at sea level ambient pressure is less than in a vacuum. This may be thought of as a change in specific impulse. The ratio of sea level to vacuum specific impulse is dependent upon the chamber pressure, nozzle area expansion ratio, and ratio of specific heats for the combustion products. Thrust coefficient tables are readily available to provide this information. It was again assumed that the greater portion of the losses would occur early in flight, and all losses were computed for vertical trajectories. The results are given in Figure 5 where K_a is plotted as a function of I_s/I.

Accuracy of Results

Accuracies to within 150 feet per second should be expected with the above results. Occasionally, cases may exist which exceed these limits. First, drag curves may not actually be similar to the one selected for this study. Secondly, simplification of the results to a form which will facilitate rapid computation has necessitated several approximations. It is believed that the results as presented will be more useful in preliminary design than extremely accurate results. Guessing that the typical first stage is designed to achieve about 10,000 feet per second, the accuracy of 150 feet per second amounts to 1.5 per cent.

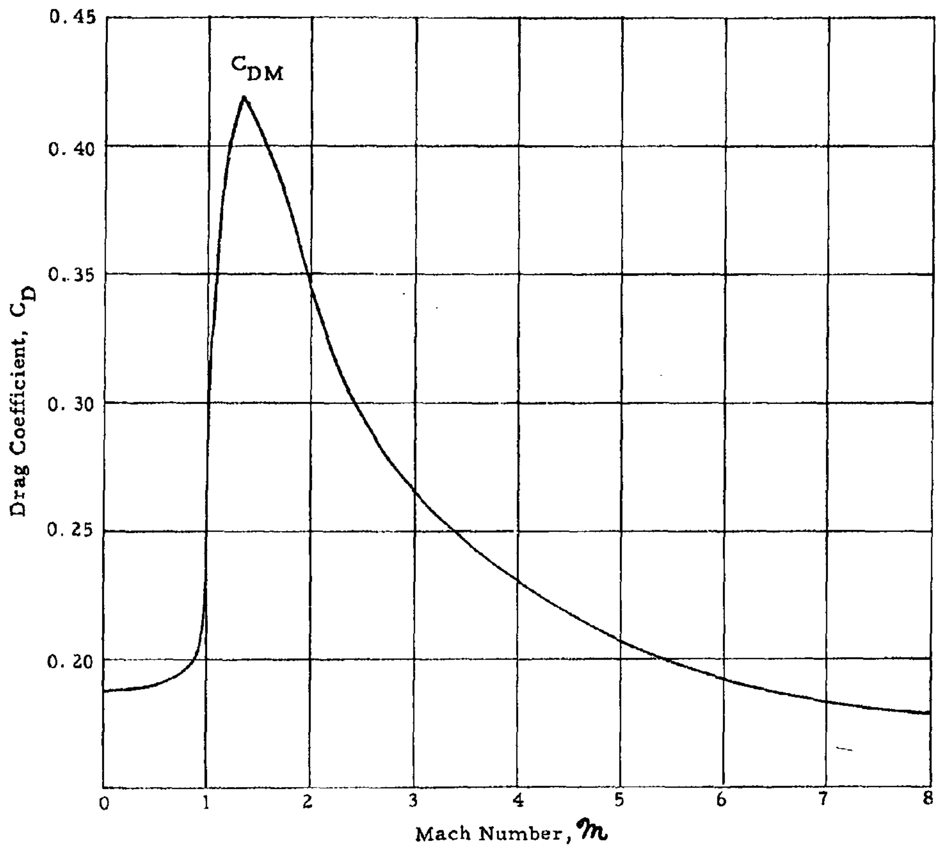

Figure 3. Drag Coefficient, C_D, Versus Mach Number.

(2013)

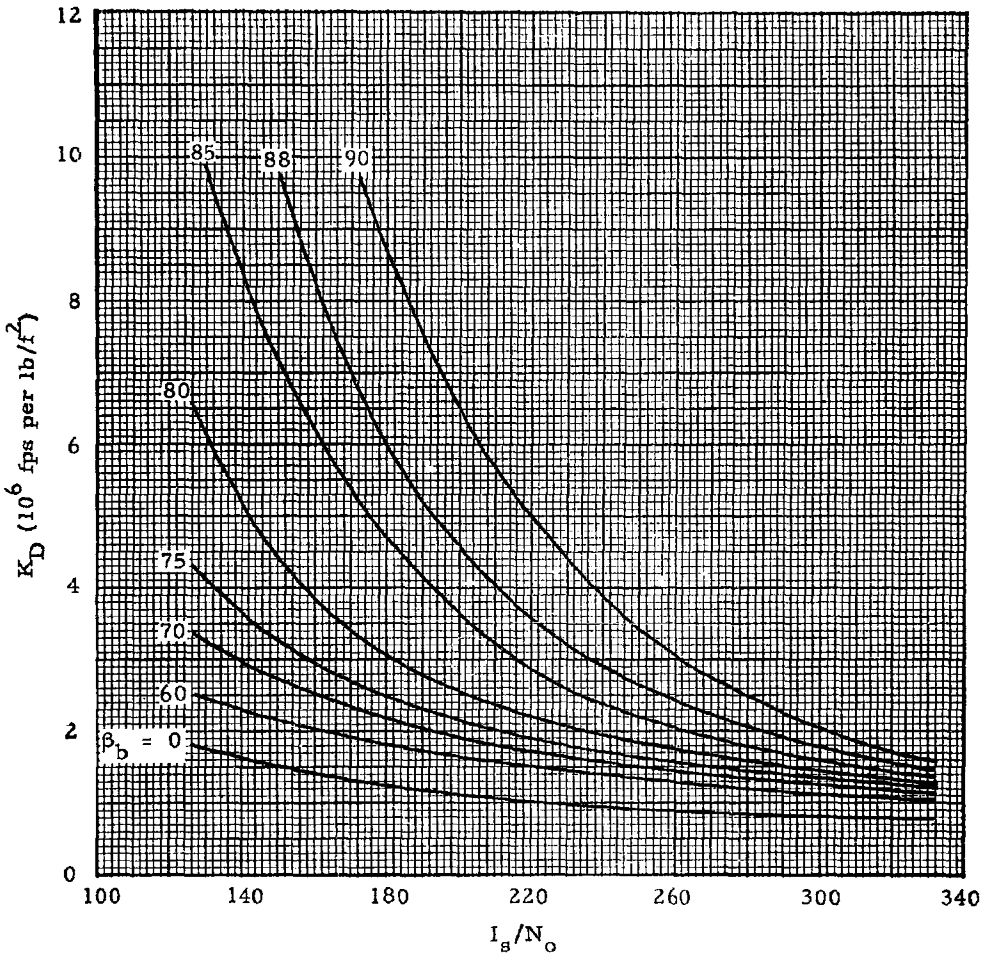

Figure 4. K_D Versus Burnout Velocity Angle, β_b, and Ratio at Sea-Level Specific Impulse to Initial Thrust-to-Weight Ratio, I_s/N_o.

(2008)

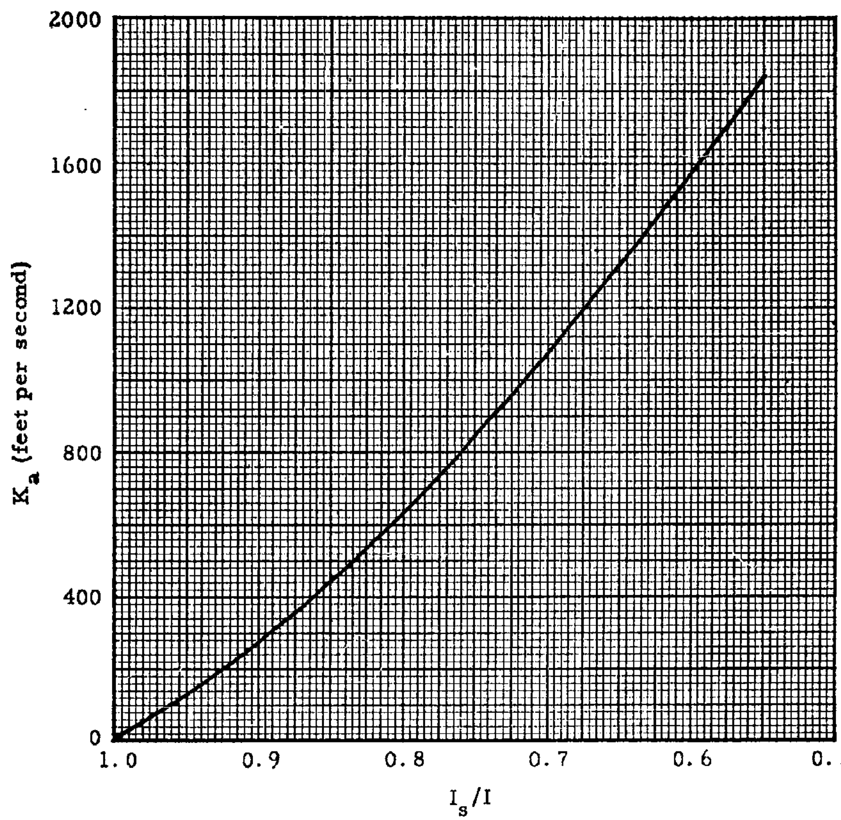

Figure 5. K_a Versus Ratio of Sea-Level to Vacuum Specific Impulse, I_s/I.

(2014)

Application to More Than One Stage

All computations were performed for single-stage vehicles, but the results may be applied to multistage vehicles.

If the first stage can be assumed to burn out at greater than 200,000 feet at a velocity angle less than 75 degrees, the drag losses may be assumed to have occurred during first stage. It is important to note that the constant K_D will be determined on the basis of the velocity burnout angle for the first stage. For multistage vehicles, this may be 5 to 15 degrees less than the angle at final-stage burnout; but for β_b less than 75 degrees, the drag losses are relatively insensitive to β_b and any reasonable estimate will probably be satisfactory.

Under almost any circumstances, the nozzle-pressure loss can be considered to occur during the first stage. Constants applicable to the first stage should be used.

The most significant velocity loss from succeeding stages will be gravitational loss. Since the velocity angle will be more constant during succeeding stages, it is usually satisfactory to assume a constant value between the assumed burnout of the first stage and the desired final burnout angle. Then the velocity loss for succeeding stages may be computed by

$$V_{L2} = g\left(\frac{R_e}{R_e+\bar{h}}\right)^2 t_{b2} \cos \bar{\beta} \tag{8}$$

where $\bar{\beta}$ is an intermediate velocity angle, $\bar{h}$ is an "average" altitude for second-stage powered flight, and the subscript 2 refers to succeeding stages. The difference between burnout angles of the first stage and that for the final burnout will depend upon the thrust pitch program selected for succeeding stages. Several authors have discussed the optimum pitch program for a variety of missions assuming powered flight in a vacuum (see references). For a ballistic missile, where impact range is the desired result, holding the thrust vector constant with respect to a stationary inertial coordinate system has been found to yield greater ranges than the gravity turn. For this case, the change in β

from first-stage burnout to final burnout will be comparatively small. In contrast, many satellite missions require that burnout angles approach or equal 90 degrees. Under these circumstances, a gravity turn or one in which the vehicle is pitched downward is a more likely trajectory. The resulting difference in burnout angles between first and final stages will be quite large.

In any trajectory in which thrust is not aligned with velocity, some energy will be expended in "turning" the velocity vector. The proportion of the thrust which goes to increasing the velocity varies as the cosine of the angle of the attack, so for small angles of attack the loss will be small.

Effect of the Earth's Rotation

The significant parameter in determining performance is the inertial velocity. Thus, the velocity of the launch point must be considered in any realistic calculation. A simple, albeit approximate, correction may be made by vectorially adding the inertial velocity vector of the launch point to the vehicle velocity vector at burnout. In several comparisons between this approximate technique and that of a machine trajectory for an eastward launch on a rotating earth, this approximation underestimated the actual burnout velocity. It has not been determined whether this is generally true; but based upon the few comparisons, we would expect the approximation to tend toward conservative results.

BURNOUT ALTITUDE VERSUS BURNOUT ANGLE

The burnout altitude is a particularly important parameter in determining payload capabilities for low-altitude satellites with circular orbits. As with the rocket equation, a closed-form expression may be derived for the distance traversed by an ideal rocket in vertical flight (constant g, no drag, constant specific impulse).

$$h^* = gI\,t_b\left(1 - \frac{\ln r}{r-1}\right) - \frac{gt_b^2}{2} \tag{9}$$

It was found that the above form could be modified to account for drag, nozzle pressure, and burnout angle.

$$h_b = \left\{h^* - \frac{(V_D+V_a)\, t_b}{2}\right\}\left\{1 - \left(\frac{\beta}{K_h}\right)^2\right\} \tag{10}$$

where

$$K_h = 93 + \frac{28}{r}\left(1 + 5\,(2 - N_o)^2\right), \quad 1 \leq N_o \leq 2 \tag{11}$$

Equation (10) assumes that the drag and nozzle-pressure losses are averaged over the duration of flight. This is not exactly true, but the approximation has proven to be satisfactory because the correction is small. The constant K_h has been determined empirically. Accuracies for Equation (10) have been found to agree with machine calculations to about 20,000 feet.

In calculating values for multistage vehicles, Equation (10) will yield the altitude of burnout for the first stage. The additional altitude achieved during succeeding stages may be calculated using the first-stage burnout velocity as computed by Equation (4) and the following relation derived by integrating Ig ln r - gt cos β at a constant, average flight path angle, $\bar{\beta}$.

$$h_{b2} = h_{b1} + V_{b1}\, t_{b2} \cos\bar{\beta} + \left\{g\, I_2\, t_{b2}\left(1 - \frac{\ln r_2}{1-r_2}\right) - \frac{g\, t_{b2}^2 \cos\bar{\beta}}{2}\right\}\cos\bar{\beta} \tag{12}$$

where the subscripts 1 and 2 refer to the first and second stage, respectively. The above form may be extended to cover additional stages. Again, an intermediate value for the flight path angle $\bar{\beta}$ may be selected between the estimated first-stage burnout flight path angle and the desired final burnout angle.

No correction is suggested for use with a rotating earth. In several comparisons with machine trajectories assuming an eastward launch on a rotating earth, the altitude value for the nonrotating earth was approximately equal to that for the rotating earth.

BURNOUT SURFACE RANGE

The surface range at burnout may be determined by the following empirical expression.

$$x_b = 1.1\, h^* \left(\frac{\beta_b}{90}\right) \tag{13}$$

The surface range is the least important of the trajectory parameters in determining gross vehicle performance. However, it is important in that it adds to the impact range of a surface-to-surface ballistic missile. Again, no correction is offered for the rotating earth because, for reasonably short flight duration, the increased inertial velocity of the vehicle and the velocity of the launch point may be assumed to cancel. Equation (13) has been found to yield surface range at burnout within an accuracy of about 10 per cent.

For multistage vehicles, the same technique used in determining altitude may be applied.

$$x_{b2} = x_{b1} + (h_{b2} - h_{b1}) \tan \bar{\beta} \tag{14}$$

FREE-FLIGHT TRAJECTORY

The calculation of the burnout conditions of a vehicle is only an intermediate step in determining its performance. Performance is usually measured in terms of impact range, apogee altitude, or some other end condition. Since all vehicles in free-flight follow a Kepler ellipse, values for range, apogee altitude, and the like may be determined from the burnout conditions by using equations yielding these values in closed form. A number of these equations are listed in Table 1.

RANGE EQUATION

Experience in the optimization of performance of medium- and long-range missiles at STL has shown that the trajectory which consists of a short period of vertical flight followed by a gravity turn to staging and a constant attitude (thrust angle with respect to launch coordinate system) throughout subsequent stages of flight yields a near-optimum range trajectory.

In the case of a single-stage missile, the constant attitude portion of the trajectory is initiated at an altitude of approximately 150,000 feet. The velocity angle of the missile at burnout is optimized for maximum range. An examination of the trajectory equations shows that the range of a missile is determined by specifying the same vehicle design parameters investigated in the previous section. (In determining the empirical equation, however, only one value of the ratio I_s/I was used, based on a chamber pressure of 500 psi, an expansion ratio of 8, and a γ of 1.24.) This study was performed at a different time from that in the preceding section, and a slightly different drag curve was assumed, but it is not expected that the results will be significantly different for this reason.

Machine calculations were performed to determine maximum range of vehicles launched from a spherical, nonrotating earth. Here, impact range is measured from the launch point rather than from the burnout point. Computer data have been used to plot a curve which shows the quantity V^* as a function of missile range. Even with a large variation in vacuum specific impulse, varying from 200 to 1000 seconds, all of the data points fall essentially on a single curve for a given N_o and $C_{DM}A/W_o$. For any other values of N_o and $C_{DM}A/W_o$ similar results are obtained. Figure 6 shows the mean curve obtained for $N_o = 1.5$ and $C_{DM}A/W_o = 0.000265$.

The results of Figure 6 have been replotted in Figure 7 on semilog paper, together with a curve given by:

$$R = D\left(e^{V^*/Bg} - 1\right) \tag{15}$$

For ranges varying from 400 to 6000 nautical miles it can be seen that Equation (15) represents the curve obtained from the machine calculations quite accurately. We have found that the parameter "B" is very insensitive to changes in N_o and $C_{DM}A/W_o$, while the parameter "D" is fairly sensitive to such changes. The values of the parameters in Figure 6 are $D = 80$ and $B = 208$.

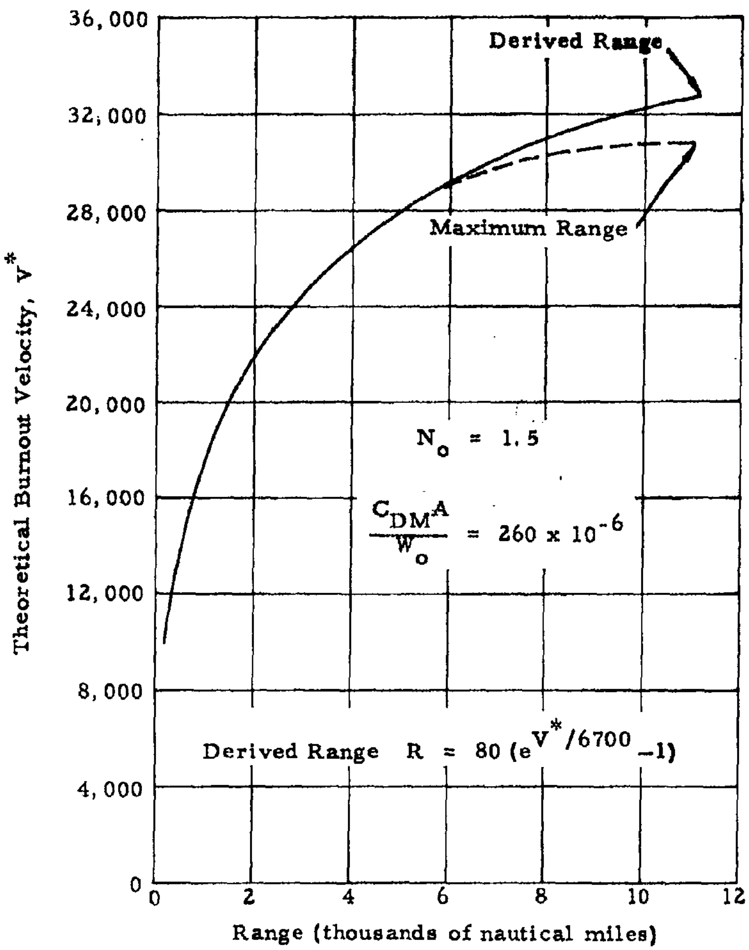

Figure 6.. Range Versus Theoretical Burnout Velocity, V^*.

(2016)

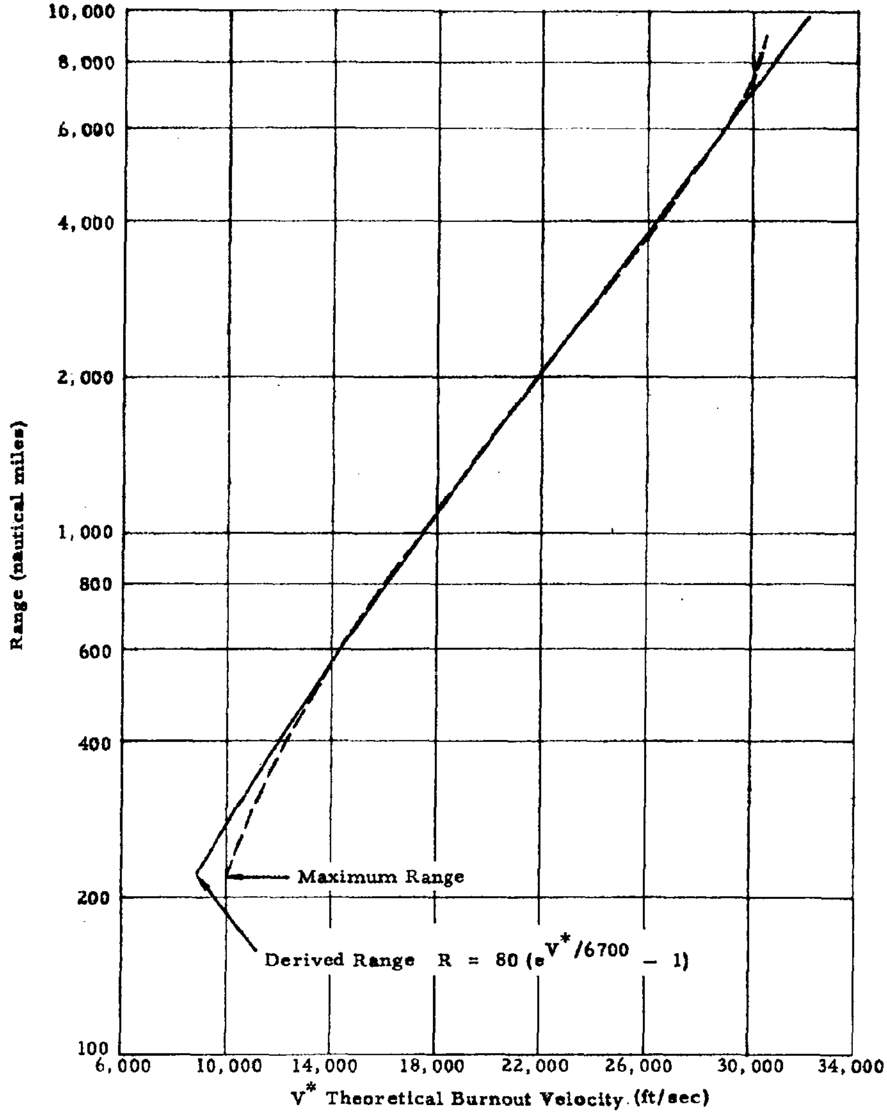

Figure 7. Range Versus Theoretical Burnout Velocity, V^*.

(2011)

The parameter B determines the slope of the fitted curve and the parameter D determines the displacement. However, the two constants must be treated as a pair. Many curves might be fitted to the empirical data, giving better accuracies in some ranges and poorer accuracies in others. We have arbitrarily selected the value of 208 seconds for B, and all values of D have been determined on this basis. If another value for B is selected, new values for D must be derived. Figure 8 shows D as a function of N_o for various values of $C_{DM}A/W_o$.

The results of Equation (15) can be extended for use from 400 to 10,800 nautical miles (halfway around the earth) by the following argument. Burnout angles were selected to maximize range. For ranges beyond 6000 nautical miles the use of maximum-range trajectories results in very large-range misses for errors in burnout speed. This can be seen by the slope of the curve in Figure 7. Lofting the trajectories so that the burnout velocity increases as determined with Equation (15) results in an increase of about five per cent above the maximum-range burnout velocity for the 10,800 nautical mile range. At the same time, the lofting decreases the miss from about 10 nautical miles to less than 2 nautical miles for an error in the burnout speed of 1 foot per second. For design purposes, it appears that deviation from the maximum-range trajectory for ranges beyond 6000 nautical miles is reasonable and, in fact, desirable.

In the case of two-stage missiles we note that

$$V^* = I_1 g \ln r_1 + I_2 g \ln r_2$$

Thus, for two-stage missiles Equation (15) becomes

$$R = D\left(r_1^{I_1/B} \, r_2^{I_2/B} - 1\right) \qquad (16)$$

By differentiating Equation (16) one may obtain a number of exchange ratios, some of which have been derived and are presented in Table 2.

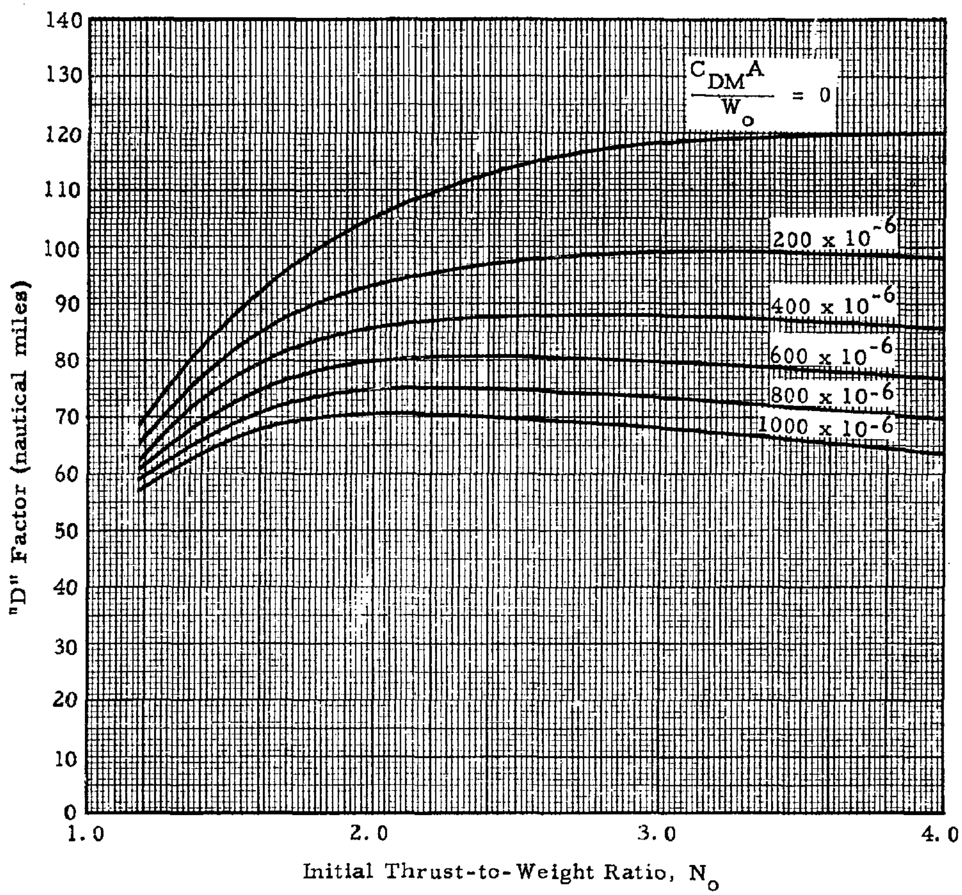

Figure 8. "D" Factor Versus Initial Thrust-to-Weight Ratio for Various $C_{DM}A/W_o$.

(2010)

Equation (16) has been checked many times against results of machine computation. To date, the equation has been accurate to about five per cent of the range. It has been found that the equation is useful in two ways. First, if the missile under study has no close counterpart and no machine data are available, a value for D as found in Figure 8 is used. Frequently, however, a vehicle is studied for which a small amount of machine data is or can be made available. In this case, the value of D is derived by solving Equation (16) "backwards." Once a value of D has been determined for the particular missile system, the calculation of perturbations of this missile system may be made using Equation (16) and the D value thus derived.

Table 1. Miscellaneous Formulas for Kepler Ellipse.

$$\sigma = \frac{R_e + h_b}{R_e} \qquad \epsilon = \sqrt{1 - 2\lambda \sin^2\beta + \lambda^2 \sin^2\beta}$$

$$\lambda = \frac{V_b^2\,\sigma_b}{g\,R_e} \qquad R_e = \text{earth radius} = 20.9 \times 10^6 \text{ feet}$$

Conservation of Energy	$V^2[t] - \frac{2\,g\,R_e^2}{z} = \text{constant}$
Conservation of Angular Momentum	$V\,z\,\sin\beta = \text{constant}$
Impact Range Angle from Burnout	$\psi = \pi - \sin^{-1}\left(\frac{1-\lambda\,\sigma\,\sin^2\beta_b}{\epsilon}\right) - \sin^{-1}\left(\frac{1-\lambda\,\sin^2\beta_b}{\epsilon}\right)$
Velocity Required to Obtain Impact Range	$V_b = \left(\frac{g\,R_e}{\sigma} \cdot \frac{1-\cos\psi}{\sigma\sin^2\beta_b + \sin\beta_b\,\sin(\psi-\beta_b)}\right)^{1/2}$
Apogee Altitude	$h_a = \frac{\sigma\,R_e\,\lambda\,\sin^2\beta_b}{1-\epsilon} - R_e$
Velocity Required to Obtain Apogee Altitude	$V_b = \left(\frac{2\,g\,R_e}{\sigma}\,\frac{1-\frac{\sigma}{\sigma_a}}{1-\left(\frac{\sigma}{\sigma_a}\sin\beta_b\right)^2}\right)^{1/2} ; \quad \sigma_a = \frac{R_e+h_a}{R_e}$
Period for Complete Elliptic Orbit	$T = 2\pi\,\frac{r_b^{3/2}}{(2-\lambda)^{3/2}\,(gR_e^2)^{1/2}}$
Time to Apogee from Burnout	$t_a = \frac{T}{2\pi}\left(\sqrt{1-\epsilon^2}\,\cot\beta_b + \cos^{-1}\left(\frac{1-\lambda}{\epsilon}\right)\right)$

Table 2. Summary of Exchange Ratios Derived from Simplified Range Equation.

Single-Stage Vehicles

R (nautical miles) $= D\left(r^{1/B} - 1\right)$

$$\frac{\partial R}{\partial V_b} = \frac{R + D}{Bg}$$

$$\left.\frac{\partial R}{\partial W_b}\right|_{W_o} = -\frac{I\,(R + D)}{B\,W_b}$$

$$\frac{\partial R}{\partial W_o} = \frac{I\,(R + D)}{B\,W_o}$$

$$\frac{\partial R}{\partial I} = \frac{(R + D)\ln r}{B}$$

$$\left.\frac{\partial W_o}{\partial W_b}\right|_{\frac{C_{DM}A}{W_o}} = r\left(1 - \frac{B}{I}\cdot\frac{RN_o}{D\,(R + D)}\cdot\frac{\partial D}{\partial N_o}\right)^{-1}$$

$$\left.\frac{\partial W_o}{\partial W_b}\right|_{N_o,\ \frac{C_{DM}A}{W_o}} = r$$

$$\frac{\partial W_o}{\partial W_L} = \frac{W_o}{W_L}$$

Two-Stage Vehicles*

$$R \text{ (nautical miles)} = D\left(r_1^{I_1/B}\, r_2^{I_2/B} - 1\right)$$

$$\frac{\partial R}{\partial V_b} = \frac{(R + D)}{Bg}$$

$$\frac{\partial R}{\partial W_j} = -\frac{I_1\,(R + D)}{B\,W_{b1}}$$

$$\frac{\partial R}{\partial W_{b2}} = -\frac{I_2\,(R + D)}{B\,W_{b2}} + \frac{\partial R}{\partial W_j}$$

$$\left.\frac{\partial R}{\partial W_{o1}}\right|_{r_1/r_2} = \frac{I_2\,(R + D)}{B\,W_{o1}}$$

$$\left.\frac{\partial R}{\partial W_{o1}}\right|_{W_{o2}} = \frac{I_1\,(R + D)}{B\,W_{o1}}$$

$$\left.\frac{\partial R}{\partial W_{o1}}\right|_{W_{o1} - W_{o2}} = \frac{I_1\,(R + D)}{B\,W_{o1}}\left(1 - r_1 + \frac{I_2\,r_1}{I_1}\right)$$

$$\frac{\partial R}{\partial I_1} = \frac{(R + D)\,\ln r_1}{B}$$

$$\frac{\partial R}{\partial I_2} = \frac{(R + D)\,\ln r_2}{B}$$

$$\left.\frac{\partial W_{o1}}{\partial W_j}\right|_{\frac{C_{DM}A}{W_o}} = r_1\left(1 - \frac{B}{I_1}\cdot\frac{R\,N_o}{D\,(R + D)}\cdot\frac{\partial D}{\partial N_o}\right)^{-1}$$

*Numerical subscripts refer to stages and are in order of burning period.

Two-Stage Vehicles (Continued)

$$\left.\frac{\partial W_{o1}}{\partial W_{b2}}\right|_{\frac{C_{DM}A}{W_o}} = r_1\, r_2 \left(1 - \frac{B}{I_2} \cdot \frac{R\, N_o}{D\,(R + D)} \cdot \frac{\partial D}{\partial N_o}\right)^{-1}$$

$$\left.\frac{\partial W_{o1}}{\partial W_j}\right|_{N_o,\ \frac{C_{DM}A}{W_o}} = r_1$$

$$\left.\frac{\partial W_{o1}}{\partial W_{b2}}\right|_{N_o,\ \frac{C_{DM}A}{W_o}} = r_1\, r_2$$

$$\frac{\partial W_{o1}}{\partial W_L} = \frac{W_{o1}}{W_L}$$

APPENDIX

This appendix reports the theoretical analysis which determined the selection of missile design parameters for this study. This analysis also suggested the use of the V_L concept in reducing the computer output data to a manageable form.

Equations of Motion

In determining the performance of a rocket, one is confronted with complicated differential equations of motion. Accurate solutions are obtained only by using a digital computer. However, one can obtain a large amount of information about such things as gravitational and atmospheric effects on the performance of boost rockets by a term-by-term examination of the equations without the computer. The basic equation of motion is:

$$\ddot{\underline{z}} = n[z, t]\,\underline{\kappa}[t] + \underline{a}[z, \dot{\underline{z}}, m] \qquad \text{(A. 1)}$$

where

$\underline{z}$ = radius vector from earth center to missile

n = thrust to mass ratio = $F[z]/m[t]$

t = time

$m[t]$ = mass of missile

$\underline{\kappa}$ = unit vector in the direction of thrust

$\underline{a}$ = $\underline{a}$[gravitation] + $\underline{a}$[drag]

$$\underline{a}\,[\text{gravitation}] = -\frac{g\,R_e^2}{z^2} \cdot \frac{\underline{z}}{z} \qquad \text{(A. 2)}$$

$$\underline{a}\,[\text{drag}] = -\frac{1}{2}\,\frac{\rho\,V^2\,C_D A}{m} \cdot \frac{\underline{V}}{V} \qquad \text{(A. 3)}$$

where

C_D = drag coefficient, a function of Mach number

R_e = radius of earth

$\underset{\sim}{V}$ = vehicle velocity relative to the atmosphere

A = reference area

ρ = air density

Replacing $\underset{\sim}{a}$ with the terms for $\underset{\sim}{a}$ [gravitation] and $\underset{\sim}{a}$ [drag] and dividing by g:

$$\frac{\ddot{\underset{\sim}{z}}}{g} = \frac{F}{W} \frac{[z]}{[t]} \underset{\sim}{K} [t] - \frac{R_e^2}{Z^2} \cdot \frac{\underset{\sim}{Z}}{Z} - \frac{1}{2} \rho V^2 \frac{C_D A}{W [t]} \cdot \frac{\underset{\sim}{V}}{V} \qquad (A.4)$$

we assume that thrust in a vacuum is proportional to the weight flow rate. Thrust as a function of altitude is taken as the vacuum thrust corrected for ambient pressure

$$F [z = \infty] = F_\infty = I \dot{W} \qquad (A.5)$$

$$F [z] = F_\infty \left(1 - \frac{p [z]}{p_s} \left(1 - \frac{I_s}{I_o}\right)\right) \qquad (A.6)$$

where

$p[z]$ = ambient pressure

p_s = ambient pressure at sea level

I_s/I = ratio of sea-level thrust to vacuum thrust for identical flow rates.

Values for I_s/I may be calculated from tables showing thrust coefficient versus expansion area ratio, ratio of specific heats for exhaust products, and chamber pressure. Defining N_o as ratio of initial thrust to initial weight and assuming constant $\dot{W}$, we can write the equation of motion in terms of missile design parameters.

$$\frac{\ddot{\underset{\sim}{z}}}{g} = N_o \frac{I}{I_s} \frac{1}{1-\frac{N_o}{I_s}t}\underset{\sim}{K} - N_o \frac{\left(\frac{I}{I_s}-1\right)\frac{p}{p_s}}{1-\frac{N_o}{I_s}t}\underset{\sim}{K} - \frac{R_e^2}{z^2} \cdot \frac{\underset{\sim}{z}}{z} - \frac{1}{2}\rho V^2 \frac{C_D A}{W_o\left(1-\frac{N_o}{I_s}t\right)} \cdot \frac{\underset{\sim}{V}}{V} \quad \text{(A. 7)}$$

In some cases, flow rate will not be constant, but we assume it to be so during the first several seconds of flight. Forming the dot product of $\underset{\sim}{V}/V$ with $\ddot{\underset{\sim}{z}}$, integrating for a gravity turn (thrust aligned with velocity) and assuming a spherical, nonrotating earth,

$$\frac{V_b}{g} = I \ln r - \int_o^{t_b} \frac{N_o\left(\frac{I}{I_s}-1\right)\frac{p}{p_s}}{1-\frac{N_o}{I_s}t} dt - \int_o^{t_b} \frac{R_e^2}{z^2}\cos\beta \, dt$$

$$- \int_o^{t_b} \frac{\frac{1}{2}\rho V^2 C_D A}{W_o\left(1-\frac{N_o}{I_s}t\right)} dt \quad \text{(A. 8)}$$

We define the velocity lost to gravitation, drag, and atmosphere as the following integrals:

$$V_g = g \int_o^{t_b} \frac{R_e^2}{z^2}\cos\beta \, dt \quad \text{(A. 9)}$$

$$V_D = g \int_0^{t_b} \frac{\frac{1}{2} \rho V^2 C_D A}{W_o \left(1 - \frac{N_o}{I_s} t\right)} dt \tag{A.10}$$

$$V_a = g \int_0^{t_b} \frac{N_o \left(\frac{I}{I_s} - 1\right) \frac{p}{p_s}}{1 - \frac{N_o}{I_s} t} dt \tag{A.11}$$

and the design velocity

$$V^* = I g \ln r .$$

So the burnout velocity becomes

$$V_b = V^* - V_g - V_D - V_a \tag{A.12}$$

It is apparent that the velocity lost is intimately tied in with the trajectory itself. Forming the dot product of $\ddot{\underline{z}}/g$ with a unit vector normal to the velocity, and again assuming a gravity turn and nonrotating earth,

$$\dot{\beta} = \frac{g}{V} \frac{R_e^2}{z^2} \sin \beta - \frac{V \sin \beta}{z} \tag{A.13}$$

For low velocity, the turning rate is large, and the greater portion of turning is to be expected early in the trajectory. However, the amount of turning to be achieved is limited by the desired burnout angle. Therefore, it is frequently necessary to keep sin β (therefore β) quite small during the early portion of the trajectory to prevent too much turning. The trajectory can be thought of as

consisting of three segments: (a) a portion during which the vehicle flies steeply, (b) a period of turning, and (c) a portion in which the velocity angle remains relatively constant.

For the early portion of flight, the velocity can be approximated by:

$$V \doteq (N_o - 1)\, g\, t \tag{A. 14}$$

For a given burnout angle, the start of the period of turning will depend primarily upon the initial thrust-to-weight ratio, N_o. Thus, for low values of N_o (near 1. 0), the initial portion of flight will be at lower velocity and the turning rate would be increased. To achieve the same burnout angle as that for a higher value of N_o, the initial portion of the trajectory must be steeper (lower β).

The turning rate for a gravity turn is zero when the vehicle velocity equals that required for a circular satellite orbit at the same altitude.

Gravity Loss

We can use the foregoing to gain insight into the behavior of the velocity lost to gravitation and atmosphere. In vertical flight, the gravity loss should be proportional to t_b. For a trajectory burning out at angle β_b, the gravity loss will be some fraction of that lost in purely vertical flight; and we would expect that fraction to depend upon N_o and the proportion of total mass consumed as propellant $(1 - 1/r)$.

It is sometimes proposed that the velocity lost to gravitation is not really lost at all but that it is converted into potential energy. It may be observed, however, that a vehicle in powered flight is not a conservative system. A ballistic missile does not burn impulsively (i. e., all the propellant is not burned on the ground). Some of the fuel is used to lift the unburned fuel, so that the vehicle always ends up at some finite altitude above the earth. Energy is imparted to the expended propellants by raising unburned propellant to some finite altitude.

One way to see what happens is to consider the following comparison of two single-stage vehicles which are identical in all respects except thrust. Figure 9 compares vertical trajectories for the two vehicles.

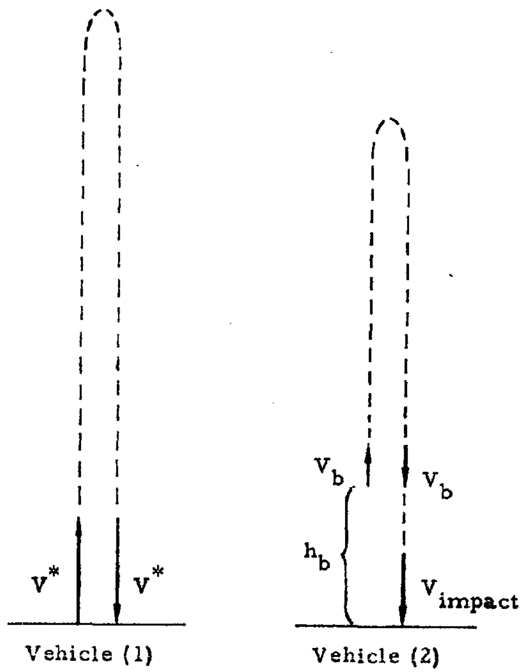

Figure 9. Comparison of Impulsive and Finite Thrust for Vertical Trajectory. (Constant gravitational field and no atmosphere.)
(2012)

With vehicle (1) we assume an infinite thrust (impulsive burning) and with vehicle (2) a finite thrust. Vehicle (1) burns out all its propellants at the surface of the earth, achieves a theoretical velocity V^*, rises, returns to earth and impacts at the same velocity. For vehicle (2),

$$V_b = V^* - g\,t_b \tag{A. 15}$$

$$V_{impact} = V^* \text{ [illegible]} \tag{A. 16}$$

$$V_{impact} = V^* - g\, t_b \left(1 - \frac{t_j}{t_b}\right) \tag{A.17}$$

where

t_b = burning time for vehicle (2)

t_j = time from burnout altitude to impact on re-entry

t_j will always be less than t_b, for it takes a time t_b to get from a velocity of 0 to V_b, whereas it takes a time t_j to get from a velocity of V_b to V_{impact}, where $V_{impact} > V_b$. The kinetic energy of vehicle (1) at impact will be essentially proportional to the square of its impact velocity, V^*. The kinetic energy of vehicle (2) will be essentially proportional to the square of its impact velocity, and V_{impact} will always be smaller than the theoretical velocity of vehicle (1). As the thrust-to-weight ratio of vehicle (2) increases, V_{impact} gets closer to V^*, and hence gravity losses decrease. In actual missiles, the thrust-to-weight ratio will be closer to one than infinity because the weights of engines and structural components will increase with increased thrust. We reach a point where the advantage of higher thrust in terms of velocity losses is offset by increase in burnout weight.

We see that not all of the velocity loss goes into gaining altitude; some is lost to the expended propellants. By substituting the appropriate numerical values for an existing vehicle into these equations, it was determined that around 25 per cent of the velocity loss went into gaining altitude; 75 per cent was lost as equivalent energy to the expended propellants. This calculation presents a good argument for holding the burnout altitude as low as possible. It is true that low-burnout altitudes mean larger drag effects, but these are relatively small when compared to gravity losses. Aerodynamic effects, of course, lead to heating, and heating often means an increase in structural weight, but gravitational losses are still a prime concern.

Drag Loss

The drag loss, Equation (A. 10), is dependent upon the ratio $C_D A/W_o$, ρ, and V^2. The air density, ρ, is dependent upon altitude and, for qualitative purposes, can be considered to exponentially decay with altitude.

$$\rho(z) = \rho_s e^{-k(z-R_e)} \tag{A. 18}$$

The dependency of the drag losses upon V^2 will be significant during late portions of flight if the trajectory is flat (low) and high velocities are achieved below, say, 150,000 feet. However, the effect of V^2 for most "normal" trajectories is not important because these values occur when the vehicle is beyond the atmosphere. The greatest erosion of velocity occurs when C_D is near its peak and early in flight when ρ is of the same magnitude as ρ_s. In a typical trajectory with an initial thrust-to-weight ratio of 1.2, the vehicle achieved Mach 1 in 80 seconds at about 30,000 feet, where the density is approximately 0.37 times that at the earth's surface.

It would be expected that for equivalent trajectories the velocity loss to drag would be sensitive to N_o. There are two effects. For high N_o higher velocities are achieved at lower altitudes, and hence the density for Mach 1 velocity is large. But for high N_o the duration of time through which the drag forces are acting is reduced, and the effects will tend to cancel. It turns out that drag losses are very insensitive to N_o.

For the most part, V_D depends upon $C_{DM}A/W_o$, N_o/I_s, and β at burnout. Because the trajectory changes little with variation in $C_{DM}A/W_o$, the losses can be expected to be proportional to this quantity. C_{DM} (the maximum C_D) is the single parameter selected to be characteristic of all drag curves for reasons stated elsewhere in this paper. The term I_s/N_o is equivalent to $W_o/\dot{W}$, which determines the change in $C_{DM}A/W[t]$ with time. For the same initial weight, the missile with lower I_s/N_o will have less weight at the time when the drag forces become most important. The burnout angle β is a

measure of the proportion of the total trajectory contained in the atmosphere. As β is increased, the density associated with each velocity is increased, and the resulting velocity loss is greater.

As the trajectory becomes very flat and high velocities are achieved at low altitudes, the effect of V^2 and the long duration of the drag force combine to increase the drag loss to very high values. It is not expected that such trajectories are realistic, as aerodynamic heating may preclude extremely flat burnout angles. Flat burnout angles may be achieved if the thrust is reduced to increase the total time of powered flight. Usually, thrust levels which are sufficient to boost the vehicle at launch yield comparatively short burning times over-all. Thrust may be reduced by throttling a single-stage vehicle or, more profitably, by staging. If either of these techniques is not sufficient, and if flat burnout velocities are required, a coast period may be inserted between burning periods. If restart capabilities are not available or not desirable, the remaining alternative is to fly the vehicle steeply during an early portion of flight and pitch down after sufficient altitude has been achieved, yielding a negative angle of attack. In this type of trajectory, considerable velocity (and payload) is lost in turning the velocity vector when the magnitude of the velocity is high. To date, no approximation has been found to determine these "turning losses"; the only realistic approach has been to use the computing machine.

Nozzle-Pressure Loss

The term V_a results from the fact that thrust is lost when the nozzle pressure in the exit plane is less than the ambient pressure. This loss is frequently thought of in terms of a reduction in specific impulse. The amount of the thrust loss as a function of trajectory parameters is dependent only upon the ambient pressure, so the total velocity loss occurs early in powered flight.

The integral shows that the nozzle-pressure loss should also be proportional to N_o. However, an increase in N_o increases the rate at which altitude is achieved, reducing the duration of flight time at high ambient pressure by an amount also dependent upon N_o; and the two effects tend to cancel. The effect of I_s/N_o, or the change in vehicle weight with time, is less significant with the nozzle-pressure loss than with the drag loss because the largest percentage of nozzle-pressure loss occurs early in powered flight.

NOMENCLATURE

A	vehicle reference area for drag calculations
B	empirical parameter in simplified range equation
C_D	drag coefficient, function of Mach number
C_{DM}	maximum drag coefficient
D	empirical parameter in simplified range equation
F	thrust (pounds)
g	gravitational constant, 32.2 feet/second^2
h	burnout altitude measured from earth's surface
$\bar{h}$	intermediate altitude between first stage burnout and final burnout to compute velocity loss in succeeding stages
h^*	burnout altitude for vertical trajectory neglecting atmospheric effects
i	index denoting stage measured from launch
I	vacuum specific impulse, vacuum thrust divided by flow rate of fuel
I_s	sea level specific impulse, sea level thrust divided by flow rate of fuel
K_a	empirical constant used to determine V_a
K_D	empirical constant used to determine V_D
K_g	empirical constant used to determine V_g
K_{gg}	empirical constant used to determine V_{gg}
$\mathcal{M}$	Mach number
m	mass of vehicle
N_o	initial thrust/weight ratio, launch thrust divided by launch weight
n	thrust/mass ratio, a function of time
p	atmospheric pressure, a function of altitude
p_s	atmospheric pressure at sea level

R	impact range
R_e	radius of earth = 20.9×10^6 feet
r	burnout mass ratio = stage initial weight (mass) divided by stage burnout weight (mass)
T	total period of elliptic orbit
t	time
t_a	time from selected trajectory conditions to apogee
t_b	burning time
t_j	time from reaching burnout altitude to impact on re-entry
$\underset{\sim}{V}$	vehicle velocity vector
$V[t]$	magnitude of velocity as function of time
V_a	velocity lost to nozzle pressure
V_b	magnitude of burnout velocity
V_D	velocity lost to drag
V_{impact}	velocity at impact
V_g	velocity lost to gravitation
V_L	total velocity lost = $V_a + V_D + V_g$
V^*	theoretical velocity as determined by rocket equation
$W[t]$	weight of vehicle, function of time
W_o	vehicle initial weight
W_b	vehicle final (burnout) weight
W_j	weight jettisoned between stages of two stage vehicle
W_p	weight of payload (includes guidance and other weights which do not vary with last stage size)
x_b	surface range at burnout

$\underset{\sim}{z}$	radius vector to vehicle from earth center
z	magnitude of $\underset{\sim}{z}$
β_b	angle between vehicle velocity vector at burnout and local vertical
$\overline{\beta}$	selected β between those for first stage and final stage burnout to be used in determining velocity losses and altitude gains
ϵ	eccentricity of free flight ellipse
$\underset{\sim}{K}$	unit vector aligned with thrust
λ	nondimensional parameter $= V_b^2\sigma/g\,R_e$
γ	ratio of specific heats of combustion products
ρ	atmospheric density, function of altitude
[ ]	brackets indicate functional notation
σ	nondimensional parameter $= R_e + h_b/R_e$
ψ	impact range angle
$\left.\frac{\partial u}{\partial v}\right\|_w$	partial derivative of u with respect to v with w held constant

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